THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4030 Differential Geometry
7 November, 2024 Tutorial

$X: U \to S_1$

$Y: V \to S_2$

Discussion at 0850

1. A pair of surfaces S_1, S_2 are called conjugate minimal surfaces if they are covered by isothermal parameterisations X, Y such that

$$X_u = -Y_v, \quad X_v = Y_u.$$

That is, they satisfy the Cauchy-Riemann equations. By PDE, this means that they are also harmonic and therefore S_1, S_2 are minimal. Show that the surface parameterised by

$$Z_t = \cos tX + \sin tY, t \in \mathbb{R}$$

is minimal by showing that Z_t is also isothermal and harmonic. This means that if we have a pair of conjugate minimal surfaces parameterised by X, Y, we can find a 1-parameter family of minimal surfaces Z_t that continuously interpolates between X $(t = 0)$ and Y $(t = \pi/2)$.

2. Compute the Christoffel symbols of a surface of revolution given by

$$X(u^1, u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2)).$$

3. Recall that a diffeomorphism $\varphi : S_1 \to S_2$ is an isometry if for any $p \in S_1$ and $v, w \in T_pS_1$,

$$\langle v, w\rangle_p = \langle d\varphi_p(v), d\varphi_p(w)\rangle_{\varphi(p)}.$$

Show that φ is an isometry if and only if the arc-length of any parameterised curve in S_1 is equal to the arc-length of the image curve under φ.

1. A pair of surfaces S_1, S_2 are called conjugate minimal surfaces if they are covered by isothermal parameterisations X, Y such that

$$X_u = -Y_v, \quad X_v = Y_u.$$

That is, they satisfy the Cauchy-Riemann equations. By PDE, this means that they are also harmonic and therefore S_1, S_2 are minimal. Show that the surface parameterised by

$$Z_t = \cos tX + \sin tY, t \in \mathbb{R}$$

is minimal by showing that Z_t is also isothermal and harmonic. This means that if we have a pair of conjugate minimal surfaces parameterised by X, Y, we can find a 1-parameter family of minimal surfaces Z_t that continuously interpolates between X $(t = 0)$ and Y $(t = \pi/2)$.

Given: $X : U \to S_1$ $\quad$ Recall isothermal means

$Y : V \to S_2$

$$[g_{uv}]_X = \begin{bmatrix} \lambda^2(u,v) & 0 \\ 0 & \lambda^2(u,v) \end{bmatrix}.$$

$$[g_{\tilde{u}\tilde{v}}]_Y = \begin{bmatrix} \gamma^2(\tilde{u},\tilde{v}) & 0 \\ 0 & \gamma^2(\tilde{u},\tilde{v}) \end{bmatrix}.$$

If X are isothermal coordinates covering S and X is harmonic,

$$\Delta X = X_{uu} + X_{vv} = 0$$

then S is minimal. (i.e. $H_S \equiv 0$).

Setup: $X_u = -Y_v$, $X_v = Y_u$. Cauchy Riemann equations

Pf: Z_t isothermic: $Z_u = \cos t X_u + \sin t Y_u$, $Z_v = \cos t X_v + \sin t Y_v$.

$$\begin{aligned}\langle Z_u, Z_u\rangle &= \langle \cos t X_u + \sin t Y_u, \cos t X_u + \sin t Y_u\rangle \\ &= \cos^2 t \langle X_u, X_u\rangle + 2\cos t \sin t\langle X_u, Y_u\rangle + \sin^2 t\langle Y_u, Y_u\rangle \\ &= \cos^2 t\,\lambda^2 + 2\cos t\sin t \langle \cancel{X_u}, \cancel{X_v}\rangle^{0} + \sin^2 t\,\lambda^2 \\ &= \lambda^2.\end{aligned}$$

$\langle Z_v, Z_v\rangle = \lambda^2$ similarly.

$$\begin{aligned}\langle Z_u, Z_v\rangle &= \langle \cos t X_u + \sin t Y_u, \cos t X_v + \sin t Y_v\rangle \\ &= \cos^2 t \cancel{\langle X_u, X_v\rangle}^{0} + \sin t\cos t\langle X_u, Y_v\rangle + \sin t \cos t\langle Y_u, X_v\rangle \\ &\quad + \sin^2 t \cancel{\langle Y_u, Y_v\rangle}^{0}\end{aligned}$$

$= -\cos t \sin t \langle X_u, X_u \rangle + \cos t \sin t \langle X_v, X_v \rangle$

$= \lambda^2 (\cos t \sin t - \cos t \sin t) = 0.$

So $[g_{u,v}]_Z = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix}$ and Z is isothermic.

$\Delta Z = Z_{uu} + Z_{vv} = \cos t X_{uu} + \sin t Y_{uu} + \cos t X_{vv} + \sin t Y_{vv}$

$= \cos t \Delta X + \sin t \Delta Y$

$= 0.$

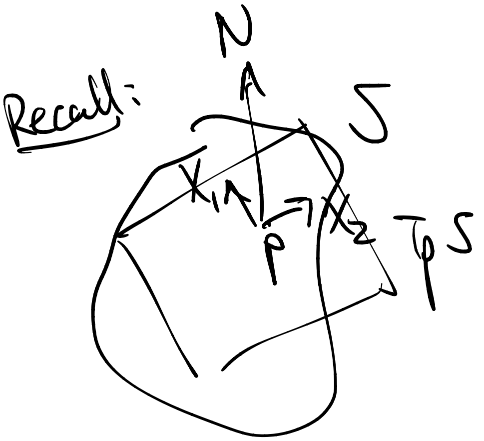
Helicoid and Catenoid can be shown to be conjugate minimal surfaces.

"associate family" of minimal surfaces.

2. Compute the Christoffel symbols of a surface of revolution given by

$$X(u^1, u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2)).$$

Recall:



So $\mathbb{R}^3 = \text{span}\{X_1, X_2, N\}$.

So regarding $\partial_i\partial_j X$ as a vector in $\mathbb{R}^3$,

$$\partial_i\partial_j X = \Gamma^1_{ij} X_1 + \Gamma^2_{ij} X_2 + \underbrace{h_{ij}}_{2^{nd}\text{ f.f.}} N.$$

$$= \sum_{k=1}^{2} \Gamma^k_{ij} X_k + h_{ij} N,$$

Einstein summation

$$= \Gamma^k_{ij} X_k + h_{ij} N$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

Pf: 1^{st} f.f. of surface of revolution

$$g = \begin{bmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{bmatrix} \quad g^{-1} = \frac{1}{f^2((f')^2 + (g')^2)} \begin{bmatrix} (f')^2 + (g')^2 & 0 \\ 0 & f^2 \end{bmatrix}.$$

Since g^{-1} is diagonal, we have

$$\Gamma^1_{11} = \frac{1}{2}\sum_{l=1}^{2} g^{1l}\left(\partial_1 g_{1l} + \partial_1 g_{1l} - \partial_l g_{11}\right)$$

$$= \frac{1}{2} g^{11}\partial_1 g_{11} + \frac{1}{2}\cancelto{0}{g^{12}}\left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right)$$

$$= \frac{1}{2} g^{11}\cancelto{0}{\partial_1 g_{11}}$$

∂_1 means $\frac{\partial}{\partial u^1}$

Since $g_{11} = f^2(u^2)$ which does not depend on u^1!

$$\Gamma^2_{11} = \frac{1}{2} g^{22} \left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11} \right) = -\frac{1}{2} g^{22} \partial_2 g_{11} = \frac{-1}{\cancel{2}} g^{22} \left(\cancel{2} f f' \right)$$

$$= \frac{-f f'}{(f')^2 + (g')^2}.$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} \left(\partial_1 \cancel{g_{21}}^{\,0} + \partial_2 g_{11} - \partial_1 \cancel{g_{12}}^{\,0} \right) = \frac{1}{2} g^{11} \partial_2 g_{11} = g^{11} f f'$$

$$= \frac{f f' \cancel{\left((f')^2 + (g')^2 \right)}}{f^2 \cancel{\left((f')^2 + (g')^2 \right)}} = \frac{f'}{f}.$$

$$\Gamma^2_{12} = 0, \quad \Gamma^1_{22} = 0, \quad \Gamma^2_{22} = \frac{f' f'' + g' g''}{(f')^2 + (g')^2}.$$

Compute K from Γ^k_{ij}, g_{ij}

3. Recall that a diffeomorphism $\varphi : S_1 \to S_2$ is an isometry if for any $p \in S_1$ and $v, w \in T_pS_1$,

$$\langle v, w\rangle_p = \langle d\varphi_p(v), d\varphi_p(w)\rangle_{\varphi(p)}.$$

Show that φ is an isometry if and only if the arc-length of any parameterised curve in S_1 is equal to the arc-length of the image curve under φ.

Pf: $\Rightarrow$: Let $\alpha : [a,b] \to S_1$ be a curve. $\varphi(\alpha)$ is a curve in S_2.

$$L\left(\varphi(\alpha)\big|_{[a,b]}\right) = \int_a^b |(\varphi \circ \alpha)'(t)|\,dt$$

$$= \int_a^b \langle d\varphi_{\alpha(t)}(\alpha'(t)), d\varphi_{\alpha(t)}(\alpha'(t))\rangle^{1/2} dt$$

$$= \int_a^b \langle \alpha'(t), \alpha'(t)\rangle^{1/2} dt$$

$$= L\left(\alpha\big|_{[a,b]}\right).$$